2.3 Calculating Limits Using the Limit Laws

Limit Laws: Suppose that c is a constant and the limits

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ **Sum Law**
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ **Difference Law**
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ **Constant Multiple Law**
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \bullet \lim_{x\to a} g(x)$ **Product Law**
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ **Quotient Law**
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ **Power Law**
7. $\lim_{x\to a} c = c$ (The limit of a constant is a constant.)
8. $\lim_{x\to a} x = a$ (The limit as x approaches a = a.)
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If n is even, we assume that a>0.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, **Root Law**

n is a positive integer. (If n is even, we assume that a>0.)

Example: Evaluate the limits and justify each step by indicating the appropriate limit law(s).

(a) $\lim_{x\to 3}(5x^3-3x^2+x-6) = \lim_{x\to 3} 5x^3 - \lim_{x\to 3} 3x^2 + \lim_{x\to 3} x - \lim_{x\to 3} 6$ (1, 2)

$= 5\lim_{x\to 3} x^3 - 3\lim_{x\to 3} x^2 + \lim_{x\to 3} x - \lim_{x\to 3} 6$ (3)

$= 5(3)^3 - 3(3)^2 + 3 - 6$ (7, 8, 9)

$= 5(27) - 3(9) + 3 - 6$

$= 135 - 27 = 3 - 6$

$\mathbf{= 105}$

(b) $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2} x^3+2x^2-1}{\lim_{x\to -2} 5-3x}$ (5)

$= \frac{\lim_{x\to -2} x^3 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - \lim_{x\to -2} 3x}$ (1, 2)

$= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}$ (3)

$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$ (7, 8, 9)

$\mathbf{= -\frac{1}{11}}$

This leads us into the next property of limits:

The Direct Substitution Property: If $\boldsymbol{f}$ is a polynomial or rational function and $\boldsymbol{a}$ is in the domain of the function $\boldsymbol{f}$, then

$$\lim_{x\to a} f(x) = f(a)$$

Example: find $\lim_{x\to 3} \frac{x^2-9}{x-3}$ **Notice** that this function is not defined at $x = 3$, therefore we cannot use the direct substitution property immediately. But we can simplify the function using algebra.

$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3}{(x-3)} = x+3$ So now we can rewrite the problem as follows:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$$

NOTE: We were able to compute the limit by replacing the given function $f(x) = \frac{x^2-9}{x-3}$ with $g(x) = x+3$, with the same limit because $f(x) = g(x)$ everywhere except when $x = 3$. Since we are only concerned with what happens as $\boldsymbol{x}$ approaches 3, and we do not compute what happens at $x = 3$, then this is okay.

Definition: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limit exists.

Example: Find $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$ Notice that if we used direct substitution, the denominator would = 0.

So we need to write $\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$ in a different form so that we can use direct substitution. To do this we need to **rationalize** the numerator.

$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \bullet \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} = \lim_{t\to 0} \frac{(1+t)-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t\to 0} \frac{1+t-1+t}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t\to 0} \frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})} =$

$\lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}}$ (Now we can use direction substitution) $= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = \mathbf{1}.$

So . . . sometimes we need to manipulate difficult functions into functions that allow us to use direct substitution.

Next – let's discuss three theorems that will help us find limits of particular kids of functions at specific points.

$\boxed{1}$ **Theorem:** $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a-} f(x) = L = \lim_{x\to a+} f(x)$ (we saw this earlier as a definition and now is it a theorem)

$\boxed{2}$ **Theorem:** If $f(x) \leq g(x)$ when $\boldsymbol{x}$ is near $\boldsymbol{a}$ (except possibly at $\boldsymbol{a}$) and the limits of $\boldsymbol{f}$ and $\boldsymbol{g}$ both exist as x approaches a, then:

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

3 The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when $\boldsymbol{x}$ is near $\boldsymbol{a}$ (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \quad \text{then} \quad \lim_{x \to a} g(x) = L$$

Example: Use The Squeeze Theorem to show that $\lim_{x \to 0}[x^2 \cos(20\pi x)] = 0$

To illustrate The Squeeze Theorem let $g(x) = x^2\cos(20\pi x)$ and find an *f(x)* that is $\leq g(x)$ and an *h(x)* that is $\geq g(x)$. Let $f(x) = -x^2$ *and* $h(x) = x^2$. If you graph all three functions on the same coordinate plane you will see the following:

As you can see, g(x) is "squeezed" between h(x) and f(x). Since the limits of h(x) and f(x) are known to be = 0, then by The Squeeze Theorem, the limit of g(x) as x approaches zero = 0.